

# Effective $su_q(2)$ models and polynomial algebras for fermion-boson Hamiltonians

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## Abstract

Schematic  $su(2) \oplus h_3$  interaction Hamiltonians, where  $su(2)$  plays the role of the pseudo-spin algebra of fermion operators and  $h_3$  is the Heisenberg algebra for bosons, are shown to be closely related to certain nonlinear models defined on a single quantum algebra  $su_q(2)$  of quasifermions. In particular,  $su_q(2)$  analogues of the Da Providencia-Schütte and extended Lipkin models are presented. The connection between  $q$  and the physical parameters of the fermion-boson system is analysed, and the integrability properties of the interaction Hamiltonians are discussed by using polynomial algebras.

## 1 Introduction

The aim of this contribution is to discuss the equivalence between systems of interacting fermions and bosons and systems of  $q$ -deformed effective fermions. The schematic fermion-boson interaction Hamiltonians that we shall deal

with are of two different types:

$$H_{++} = \omega_f(T_0 + \Omega) + \omega_b B^\dagger B + G \left( T_+^k (B^\dagger)^p + T_-^k (B)^p \right) \quad (1.1)$$

$$H_{+-} = \omega_f(T_0 + \Omega) + \omega_b B^\dagger B + G \left( T_+^k (B)^p + T_-^k (B^\dagger)^p \right) \quad (1.2)$$

and in this paper we shall restrict our study to the cases with  $k, m = 1, 2$ . The (two-level) fermions are represented by the collective pseudospin operators  $\{T_0, T_\pm\}$  and the energy difference between fermion levels is fixed by the energy scale  $\omega_f$ . Fermions are coupled to an external boson field with frequency  $\omega_b$  that is quantized through the creation (annihilation) operators  $B^\dagger (B)$ .

Many interesting quantum models fall into this category of Hamiltonians. In particular,  $H_{+-}$  with  $k = p = 1$  is the well-known Dicke model in Quantum Optics [1], that exhibits many of the characteristic features of quantum nonlinear phenomena [2]. Models based on the coupling between bi-fermions and bosons have been introduced long ago [3, 4] (see also the review paper [5] and references therein) and are particularly suitable to describe condensation phenomena and transitions from fermionic to bosonic phases. Among them, we shall study the Da Providencia-Schütte model [6] (that corresponds to the case  $H_{++}$  with  $k = p = 1$ ) and two different extensions ( $k = 2$  and  $p = 1$ ) [7] of the Lipkin-Meshkov-Glick model ( $k = 2$  and  $p = 0$ ) [7, 8].

It turns out that all the abovementioned models can be replaced by effective  $su_q(2)$  quasifermion Hamiltonians with no boson operators, provided the deformation parameter  $q$  is fitted in a suitable way in terms of physical constraints. We will refer to numerical analysis that strongly confirm this statement [9] and we shall comment on some of the integrability properties of this class of interactions.

In this respect, we recall that the exact solvability of the  $su_q(2)$  interaction Hamiltonian

$$H_q^{\text{int}} = q^{\frac{\tilde{T}_0}{2}} (\tilde{T}_+ + \tilde{T}_-) q^{\frac{\tilde{T}_0}{2}} \quad (1.3)$$

was already found in [10], and this operator will be used as the building block for the effective  $q$ -Hamiltonians that we are going to introduce. Explicitly, the eigenvalues of (1.3) are just the  $q$ -numbers  $[2m]_q$  ( $m = -j, \dots, +j$ ), and the associated eigenvectors can be expressed in terms of  $q$ -Krawtchouk polynomials [11]. On the other hand, the use of quantum deformations of  $su(2)$  in this context is quite natural from a purely algebraic point of view,

since it is well known that fermion-boson models are related to polynomial generalizations of the  $su(2)$  algebra (see, for instance, [12]). In particular, if we consider the operators

$$K_0 = T_0 \quad K_+ = T_+^k (B^\dagger)^p \quad K_- = T_-^k (B)^p \quad (1.4)$$

and we take a representation of them on a certain invariant subspace of the interaction Hamiltonian, we get a commutation rule of the type

$$[K_+, K_-] = F(K_0) \quad (1.5)$$

where  $F(K_0)$  is a polynomial of  $K_0$  with degree  $(2k + p - 1)$ , and the same is true if we consider the operators  $K_+ = T_+^k (B)^p$  and  $K_- = T_-^k (B^\dagger)^p$ .

In this paper, we shall use this polynomial algebra approach to get new (to our knowledge) integrals of the motion for all the models under study. Moreover, since the coefficients of the polynomial  $F(K_0)$  are given in terms of the physical parameters of the model (degeneracy of the fermion shells, quantum numbers of the invariant subspace, etc.) we will be able to obtain new interpretations of the transition to certain dynamical regimes (for instance, the strong field limit [2]) in terms of algebraic transformations like contraction processes [13]. In general, we expect that the comparison between the algebraic properties of such polynomial algebras and those of  $su_q(2)$  will explain in more fundamental terms the efficiency of quantum algebras in order to model effective fermion-boson interactions.

## 2 The Da Providencia-Schütte model

The model proposed by Da Providencia and Schütte (DPS) is a solvable model which exhibits a phase transition between nucleonic and pionic condensates and consists of  $N = 2\Omega$  fermions moving in two single-shells each with degeneracy  $2\Omega$ . The DPS Hamiltonian reads [6]

$$H = \omega_f(T_0 + \Omega) + \omega_b B^\dagger B + G(T_+ B^\dagger + T_- B), \quad (2.1)$$

where  $G$  is the strength of the interaction and  $\{T_0, T_\pm\}$  are the generators of the  $su(2)$  algebra of collective fermions:

$$[T_0, T_+] = T_+, \quad [T_0, T_-] = -T_-, \quad [T_+, T_-] = 2T_0. \quad (2.2)$$

The Hamiltonian (2.1) commutes with the operator

$$P = B^\dagger B - (T_0 + \Omega). \quad (2.3)$$

Therefore, the matrix elements of  $H$  can be computed in a basis  $|m_\Omega, n\rangle$  labeled by the eigenvalues of the number operators for fermions and bosons. In this basis the eigenvalues of  $P$  are given by

$$P|m_\Omega, n\rangle = (n - m_\Omega - \Omega)|m_\Omega, n\rangle. \quad (2.4)$$

In particular, we shall diagonalize  $H$  in the subspace spanned by the states  $|m_\Omega, L + m_\Omega + \Omega\rangle \equiv |m_\Omega; L, \Omega\rangle$  which have a fixed eigenvalue  $L$  of  $P$

$$P|m_\Omega; L, \Omega\rangle = L|m_\Omega; L, \Omega\rangle. \quad (2.5)$$

In this subspace, the non-zero matrix elements of  $H$  are [9]

$$\langle m_\Omega; L, \Omega | H | m_\Omega; L, \Omega \rangle = \omega_b L + (\omega_f + \omega_b)(\Omega + m_\Omega), \quad (2.6)$$

$$\langle m_\Omega + 1; L, \Omega | H | m_\Omega; L, \Omega \rangle = \frac{G\sqrt{(\Omega + m_\Omega + 1)(\Omega - m_\Omega)(L + \Omega + m_\Omega + 1)}}{G\sqrt{(\Omega + m_\Omega + 1)(\Omega - m_\Omega)(L + \Omega + m_\Omega + 1)}}. \quad (2.7)$$

The dimension of such subspace depends on the sign of  $L$ . For  $L \geq 0$  the quantum number  $m_\Omega$  can take the values

$$m_\Omega = -\Omega, -\Omega + 1, \dots, \Omega, \quad (2.8)$$

and the subspace has dimension  $2\Omega + 1$ . If  $L < 0$ ,  $m_\Omega$  takes the values

$$m_\Omega = -L - \Omega, -L - \Omega + 1, \dots, \Omega, \quad (2.9)$$

and accordingly, the dimension of the invariant subspace is  $2\Omega + L + 1$ .

In general, we shall say that the system is in a *normal* phase when the correlated ground state is the eigenstate of the symmetry operator  $P$  with the eigenvalue  $L = 0$ . The denomination *deformed* phase will be assigned to cases where the correlated ground state is an eigenstate of  $P$  with eigenvalue  $L \neq 0$ : if  $L > 0$  we shall have a so-called *bosonic* phase and the case  $L < 0$  corresponds to a *fermionic* one.

## 2.1 The DPS algebra

Let us define the operators

$$K_0 = T_0 \quad K_+ = T_+ B^\dagger \quad K_- = T_- B. \quad (2.10)$$

For a given  $\Omega$  and  $L \geq 0$ , if we consider the action of these operators within the subspace  $|m_\Omega; L, \Omega\rangle$  the following DPS algebra is obtained:

$$\begin{aligned} [K_0, K_\pm] &= \pm K_\pm \\ [K_+, K_-] &= -\Omega(\Omega + 1) + (1 + 2L + 2\Omega) K_0 + 3K_0^2. \end{aligned} \quad (2.11)$$

It can also be checked that the DPS algebra for the  $L < 0$  case is just (2.11) where  $L$  is replaced by  $-L$ . Therefore, a new integral of the motion for the DPS model is given by the Casimir operator of the DPS-algebra, that can be found by standard methods [14] and reads

$$C = K_+ K_- + K_0^3 + (L + \Omega - 1) K_0^2 - \{(L + \Omega) + \Omega(\Omega + 1)\} K_0. \quad (2.12)$$

In the corresponding  $L \geq 0$  subspace, the eigenvalue of  $C$  is

$$\Omega(\Omega + 1)(\Omega + L). \quad (2.13)$$

We remark that the DPS algebra (2.11) is a quadratic generalization of the  $su(2)$  Lie algebra. In fact, the  $su(2)$  algebra can be obtained as a contraction [13] of the DPS algebra in the  $L \rightarrow \infty$  limit. Namely, if we define the ‘‘contracted generators’’ as

$$J_0 = K_0 \quad J_\pm = \frac{1}{\sqrt{L}} K_\pm \quad (2.14)$$

and we compute their commutation rules we get

$$\begin{aligned} [J_0, J_\pm] &= \pm J_\pm \\ [J_+, J_-] &= \frac{1}{L} \{-\Omega(\Omega + 1) + (1 + 2L + 2\Omega) J_0 + 3J_0^2\} \end{aligned} \quad (2.15)$$

which is still isomorphic to the DPS algebra. However, the  $L \rightarrow \infty$  limit of these commutation rules gives

$$[J_0, J_\pm] = \pm J_\pm \quad [J_+, J_-] = 2J_0. \quad (2.16)$$

In physical terms, the  $L \rightarrow \infty$  limit is just the well-known “strong field” limit of the Dicke model in Quantum Optics [2], for which the interaction dynamics is given by  $su(2)$ . If we rewrite the DPS Casimir in terms of the contracted generators we find

$$C = L J_+ J_- + J_0^3 + (L + \Omega - 1) J_0^2 - \{(L + \Omega) + \Omega(\Omega + 1)\} J_0 \quad (2.17)$$

and the  $su(2)$  Casimir operator  $C_{su(2)}$  is obtained by computing

$$C_{su(2)} = \lim_{L \rightarrow \infty} \frac{C}{L} = J_+ J_- + J_0(J_0 - 1). \quad (2.18)$$

Note that the contracted eigenvalues are just  $\Omega(\Omega + 1)$ , as it should be. As we shall see in the sequel, this polynomial algebra approach can be applied to all the fermion-boson Hamiltonians under consideration.

## 2.2 Effective $su_q(2)$ Hamiltonians for the DPS model

The quantum algebra  $su_q(2)$  is a Hopf algebra deformation of  $su(2)$  [15] with generators  $\{\tilde{T}_\pm, \tilde{T}_0\}$  and commutation rules

$$[\tilde{T}_0, \tilde{T}_\pm] = \pm \tilde{T}_\pm, \quad [\tilde{T}_+, \tilde{T}_-] = [2 \tilde{T}_0]_q, \quad (2.19)$$

where the  $q$ -number  $[x]_q$  is defined by

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}} = \frac{\sinh(z x)}{\sinh(z)}. \quad (2.20)$$

We shall use alternatively  $q$  and  $z$  (where  $q = e^z$ ) as the deformation parameter, and we shall assume that  $q$  is real. The  $su(2)$  algebra (2.2) is recovered from (2.19) in the limit  $q \rightarrow 1$  ( $z \rightarrow 0$ ).

When  $q$  is not a root of unity, the irreducible representations of  $su_q(2)$  are obtained as a straightforward generalization of those of  $su(2)$  [16, 17]:

$$\begin{aligned} \tilde{T}_0 |j, m\rangle &= m |j, m\rangle, \\ \tilde{T}_+ |j, m\rangle &= \sqrt{[j + m + 1]_q [j - m]_q} |j, m + 1\rangle, \\ \tilde{T}_- |j, m\rangle &= \sqrt{[j - m + 1]_q [j + m]_q} |j, m - 1\rangle. \end{aligned} \quad (2.21)$$

By following [9, 10], we consider an effective Hamiltonian defined as

$$H_q = \omega_b L + (\omega_b + \omega_f)(\tilde{T}_0 + \Omega) + \chi(q)q^{\frac{\tilde{T}_0}{2}}(\tilde{T}_+ + \tilde{T}_-)q^{\frac{\tilde{T}_0}{2}} \quad (2.22)$$

where  $\chi(q)$  is a scalar function to be fixed, and  $H_q$  will be realized in a  $su_q(2)$  irreducible representation with the same dimension as the subspace spanned by  $|m_\Omega; L, \Omega\rangle$  (therefore, with  $j = j(\Omega, L)$ ). The non-vanishing matrix elements of (2.22) read

$$\langle j, m | H_q | j, m \rangle = \omega_b L + (\omega_f + \omega_b)(m + \Omega), \quad (2.23)$$

$$\langle j, m + 1 | H_q | j, m \rangle = \chi(q)q^{(m+\frac{1}{2})} \sqrt{[j + m + 1]_q [j - m]_q}. \quad (2.24)$$

In order to fit the dimension of the representation with respect to the invariant subspace of the DPS model, we have to take  $j = \Omega$  and  $m = m_\Omega$  for the effective  $L \geq 0$  model, while for  $L < 0$ ,  $j = \Omega + \frac{L}{2}$  and  $m = m_\Omega + \frac{L}{2}$ .

The main conclusion of [9] (see also [10] for the Dicke model) is that the Hamiltonian (2.1) is essentially equivalent to (2.22). In other words, the bosonic degrees of freedom included in (2.1) may be reabsorbed by the  $q$ -deformation in (2.22) provided that  $q$  is defined as an appropriate function of both  $\Omega$  and  $L$ . In this way it is possible to regard  $H_q$  as an effective Hamiltonian with physical properties similar to those of  $H$ . As it is extensively shown in [9] through numerical studies, both the ground state energy and the full spectrum of the DPS model can accurately be reproduced by using the effective  $q$ -Hamiltonian (2.22).

### 2.3 The $q$ -DPS algebra

By following the same algebraic approach leading to the previous DPS algebra, now we should consider the  $su_q(2)$  operators

$$K_0 = \tilde{T}_0 \quad K_+ = q^{\frac{\tilde{T}_0}{2}} \tilde{T}_+ q^{\frac{\tilde{T}_0}{2}} \quad K_- = q^{\frac{\tilde{T}_0}{2}} \tilde{T}_- q^{\frac{\tilde{T}_0}{2}} \quad (2.25)$$

such that the effective Hamiltonian (2.22) is a linear function of  $K_0$  and  $K_\pm$ . In this new basis, the  $q$ -commutation rules of  $su_q(2)$  read

$$[K_0, K_\pm] = \pm K_\pm, \quad q K_+ K_- - q^{-1} K_- K_+ = q^{2K_0} [2 K_0]_q \quad (2.26)$$

and these expressions hold for any irreducible representation  $j$  of  $su_q(2)$ . The Casimir element for this algebra is

$$C_q = [K_0]_q [K_0 - 1]_q + q^{-2K_0+1} K_+ K_-, \quad (2.27)$$

and its eigenvalue is just  $[j]_q [j + 1]_q$ . Obviously,  $C_q$  is an integral of the motion for  $H_q$  (2.22). By working on a fixed irreducible representation  $j$ , the latter  $q$ -commutator can be rewritten as the following commutation rule (that hereafter we will call the  $q$ -DPS algebra):

$$[K_+, K_-] = \frac{q^{2j+2} + q^{-2j}}{1 - q^2} q^{2K_0} - \frac{1 + q^2}{1 - q^2} q^{4K_0}, \quad (2.28)$$

which should have algebraic properties closely related to the ones of the DPS algebra (2.11), since both models are physically equivalent. In fact, the analytic fitting  $q = q(\Omega, L)$  should be found by comparing the properties of the DPS and  $q$ -DPS algebras. Work on this open problem is actually in progress [22].

### 3 Extended Lipkin models

As a second example of Hamiltonians including fermionic and bosonic degrees of freedom, let us introduce the extended Lipkin-Meshkov-Glick Hamiltonian (LE model) [7], which is just (1.2) with  $k = 2$  and  $p = 1$ :

$$H_{+-} = \omega_f(T_0 + \Omega) + \omega_b B^\dagger B + G(T_+^2 B + T_-^2 B^\dagger). \quad (3.1)$$

which commutes with the operator [18]

$$P_{(+)} = B^\dagger B + \frac{1}{2}(T_0 + \Omega). \quad (3.2)$$

Therefore, the matrix elements of the LE model  $H_{+-}$  can be calculated in a basis labeled by the eigenvalues of  $P_{(+)}$ :

$$P_{(+)} |m_\Omega, n\rangle = L |m_\Omega, n\rangle = (n + \frac{1}{2}(\Omega + m_\Omega)) |m_\Omega, n\rangle, \quad (3.3)$$

and we shall consider the invariant subspace with  $L$  fixed:

$$|m_\Omega, L - \frac{1}{2}(\Omega + m_\Omega)\rangle \equiv |m_\Omega; L, \Omega\rangle. \quad (3.4)$$



In this subspace the non-zero matrix elements of  $H$  read

$$\langle m_\Omega; L, \Omega | H_{+-} | m_\Omega; L, \Omega \rangle = \omega_b L + (\omega_f - \frac{1}{2}\omega_b)(\Omega + m_\Omega), \quad (3.5)$$

$$\begin{aligned} \langle m_\Omega + 2; L, \Omega | H_{+-} | m_\Omega; L, \Omega \rangle = G \sqrt{L - \frac{1}{2}(\Omega + m_\Omega) \times} \\ \sqrt{(\Omega + m_\Omega + 2)(\Omega + m_\Omega + 1)(\Omega - m_\Omega)(\Omega - m_\Omega - 1)}, \end{aligned} \quad (3.6)$$

and we have to distinguish the following classes of invariant subspaces:

$$\begin{aligned} L \geq \Omega, L \text{ integer}, \quad m_\Omega + \Omega = 0, 2, \dots, 2\Omega, \\ L > \Omega, L \text{ half integer}, \quad m_\Omega + \Omega = 1, 3, \dots, 2\Omega - 1, \\ L < \Omega, L \text{ integer}, \quad m_\Omega + \Omega = 0, 2, \dots, 2L, \\ L < \Omega, L \text{ half integer}, \quad m_\Omega + \Omega = 0, 2, \dots, 2L - 1. \end{aligned} \quad (3.7)$$

Another Lipkin-type Hamiltonian can also be defined through (1.1) with  $k = 2$  and  $p = 1$ :

$$H_{++} = \omega_f(T_0 + \Omega) + \omega_b B^\dagger B + G(T_+^2 B^\dagger + T_-^2 B), \quad (3.8)$$

which differs from (3.1) in the ground state correlations [7]. Since  $H_{++}$  commutes with the operator

$$P_{(-)} = B^\dagger B - \frac{1}{2}(T_0 + \Omega), \quad (3.9)$$

its matrix elements can be calculated in a basis labeled by the eigenvalues of  $P_{(-)}$ . Namely

$$P_{(-)} | m_\Omega, n \rangle = (n - \frac{1}{2}(\Omega + m_\Omega)) | m_\Omega, n \rangle, \quad (3.10)$$

and in this case we shall compute the matrix elements within the subspace spanned by the states  $| m_\Omega, L + \frac{1}{2}(\Omega + m_\Omega) \rangle \equiv | m_\Omega; L, \Omega \rangle$ . The non-zero matrix elements of  $H_{++}$  (3.8) are now given by

$$\langle m_\Omega; L, \Omega | H_{++} | m_\Omega; L, \Omega \rangle = \omega_b L + (\omega_f + \frac{1}{2}\omega_b)(\Omega + m_\Omega), \quad (3.11)$$

$$\begin{aligned} \langle m_\Omega + 2; L, \Omega | H_{++} | m_\Omega; L, \Omega \rangle = G \sqrt{L + \frac{1}{2}(\Omega + m_\Omega) + 1 \times} \\ \sqrt{(\Omega + m_\Omega + 2)(\Omega + m_\Omega + 1)(\Omega - m_\Omega)(\Omega - m_\Omega - 1)}, \end{aligned} \quad (3.12)$$

where the dimension of the subspace depends on  $L$  and  $\Omega$ , since we have to consider the following possibilities for the set of values of the quantum number  $m_\Omega$ :

$$\begin{aligned}
L \geq 0, L \text{ integer}, \quad m_\Omega + \Omega &= 0, 2, \dots, 2\Omega, \\
L > 0, L \text{ half integer}, \quad m_\Omega + \Omega &= 1, 3, \dots, 2\Omega - 1, \\
L < 0, L \text{ integer}, \quad m_\Omega + \Omega &= -2L, -2L + 2, \dots, 2\Omega, \\
L < 0, L \text{ half integer}, \quad m_\Omega + \Omega &= -2L, -2L + 2, \dots, 2\Omega - 1.
\end{aligned} \tag{3.13}$$

### 3.1 LE algebras and their Casimir operators

The polynomial algebra approach can also be used for the LE models (3.1) and (3.8). We start our analysis by recalling the cubic algebra linked to the original Lipkin-Meshkov-Glick (LMG) Hamiltonian [8], since the latter will appear as the strong field limit for the extended LE models.

#### 3.1.1 The LMG algebra

We recall the  $su(2)$  LMG Hamiltonian [8] given by

$$H = \omega_f T_0 + \chi (T_+^2 + T_-^2). \tag{3.14}$$

If we define (see also [19])

$$K_0 = T_0 \quad K_+ = T_+^2 \quad K_- = T_-^2 \tag{3.15}$$

we get the cubic algebra

$$\begin{aligned}
[K_0, K_\pm] &= \pm 2 K_\pm \\
[K_+, K_-] &= 4\{2\Omega(\Omega + 1) - 1\} K_0 - 8 K_0^3
\end{aligned} \tag{3.16}$$

where we have identified  $j \equiv \Omega \geq 0$ . Note that this algebra is isomorphic to the Higgs algebra [20, 21] for any value of  $\Omega$ . The Casimir operator for this algebra is found to be:

$$C = K_+ K_- - K_0^4 + 4 K_0^3 + \{2\Omega(\Omega + 1) - 5\} K_0^2 - 2\{2\Omega(\Omega + 1) - 1\} K_0 \tag{3.17}$$

and the eigenvalues of this operator are  $\Omega(\Omega^2 - 1)(\Omega + 2)$ .

### 3.1.2 The $LE_{+-}$ algebra

From (3.1), we can consider the operators

$$K_0 = T_0 \quad K_+ = T_+^2 B \quad K_- = T_-^2 B^\dagger. \quad (3.18)$$

If we compute its action on an invariant subspace of the type  $L \geq \Omega$ , we obtain the following quartic generalization of the  $su(2)$  Lie algebra:

$$\begin{aligned} [K_0, K_\pm] &= \pm 2 K_\pm \\ [K_+, K_-] &= \alpha_0 + \alpha_1 K_0 + \alpha_2 K_0^2 + \alpha_3 K_0^3 + \alpha_4 K_0^4 \end{aligned} \quad (3.19)$$

whose structure constants  $\alpha_i(\Omega, L)$  read

$$\begin{aligned} \alpha_0 &= \Omega(\Omega^2 - 1)(\Omega + 2) \\ \alpha_1 &= 2(1 - 2\Omega(\Omega + 1))(\Omega - 2L - 1) \\ \alpha_2 &= 7 - 6\Omega(\Omega + 1) \\ \alpha_3 &= 4(\Omega - 2L - 1) \\ \alpha_4 &= 5. \end{aligned} \quad (3.20)$$

The Casimir operator for (3.18) can also be computed and gives a new integral of the motion for  $H_{+-}$ :

$$C = K_+ K_- + \beta_0 + \beta_1 K_0 + \beta_2 K_0^2 + \beta_3 K_0^3 + \beta_4 K_0^4 + \beta_5 K_0^5 \quad (3.21)$$

$$\begin{aligned} \text{with } \beta_0 &= -\Omega(\Omega^2 - 1)(\Omega + 2) \\ \beta_1 &= 2(L + 1) + \frac{1}{2}\Omega(-12 + \Omega(\Omega^2 + 6\Omega - 5) - 8(\Omega + 1)L) \\ \beta_2 &= -6 - 5L + \frac{1}{2}\Omega(13 + 2\Omega(3 - \Omega) + 4(\Omega + 1)L) \\ \beta_3 &= \frac{13}{2} - \Omega(\Omega + 3) + 4L \\ \beta_4 &= \frac{1}{2}\Omega - 3 - L \\ \beta_5 &= \frac{1}{2} \end{aligned}$$

The eigenvalue of  $C$  is found to be  $-\frac{1}{2}\Omega(\Omega^2 - 1)(\Omega + 2)(\Omega - 2L)$ . It can also be checked that the transformation (2.14) leads to the LMG algebra (3.16) and Casimir (3.17) as the ‘‘strong field’’ contraction  $L \rightarrow \infty$  of (3.19) and (3.21), respectively. Similar quartic algebras can be obtained for the remaining invariant subspaces [22].

### 3.1.3 The $LE_{++}$ algebra

For the second class of LE models (3.8) we define the generators

$$K_0 = T_0 \quad K_+ = T_+^2 B^\dagger \quad K_- = T_-^2 B. \quad (3.22)$$

The associated quartic algebra for  $L \geq 0$  reads:

$$\begin{aligned} [K_0, K_\pm] &= \pm 2 K_\pm \\ [K_+, K_-] &= \alpha_0 + \alpha_1 K_0 + \alpha_2 K_0^2 + \alpha_3 K_0^3 + \alpha_4 K_0^4 \\ \text{with } \alpha_0 &= -\Omega(\Omega^2 - 1)(\Omega + 2) \\ \alpha_1 &= -2(1 - 2\Omega(\Omega + 1))(\Omega + 2L + 1) \\ \alpha_2 &= -7 + 6\Omega(\Omega + 1) \\ \alpha_3 &= -4(\Omega + 2L + 1) \\ \alpha_4 &= -5 \end{aligned} \quad (3.23)$$

The Casimir operator is now

$$\begin{aligned} C &= K_+ K_- + \beta_0 + \beta_1 K_0 + \beta_2 K_0^2 + \beta_3 K_0^3 + \beta_4 K_0^4 + \beta_5 K_0^5 \\ \text{with } \beta_0 &= \Omega(\Omega^2 - 1)(\Omega + 2) \\ \beta_1 &= 2L - \frac{1}{2}\Omega(-4 + \Omega(\Omega^2 + 6\Omega + 3)) + 8(\Omega + 1)L \\ \beta_2 &= 1 - 5L + \frac{1}{2}\Omega(-9 + 2\Omega(\Omega - 1) + 4(\Omega + 1)L) \\ \beta_3 &= -\frac{5}{2} + \Omega(\Omega + 3) + 4L \\ \beta_4 &= -\frac{1}{2}\Omega + 2 - L \\ \beta_5 &= -\frac{1}{2} \end{aligned} \quad (3.24)$$

and the eigenvalue of  $C$  is  $\frac{1}{2}\Omega(\Omega^2 - 1)(\Omega + 2)(2 + \Omega + 2L)$ . Once more, the ‘‘strong field’’ contraction of this algebra gives rise to the same LMG algebra (3.16), that underlies the asymptotic  $L \rightarrow \infty$  dynamics of both LE models.

## 3.2 Effective $su_q(2)$ Hamiltonians for the LE models

Like in the case of the DPS model, an effective  $q$ -Hamiltonian for (3.1) has been introduced in [9]

$$H_q = \omega_b L + (\omega_f - \frac{1}{2}\omega_b)(\tilde{T}_0 + \Omega) + \chi(q)q^{\tilde{T}_0}(\tilde{T}_+^2 + \tilde{T}_-^2)q^{\tilde{T}_0}, \quad (3.25)$$

which has the following non-vanishing matrix elements

$$\langle j, m | H_q | j, m \rangle = \omega_b L + (\omega_f - \frac{1}{2}\omega_b)(\Omega + m), \quad (3.26)$$

$$\langle j, m + 2 | H_q | j, m \rangle = \chi(q) q^{2(m+1)} \times \sqrt{[j + m + 2]_q [j + m + 1]_q [j - m]_q [j - m - 1]_q}. \quad (3.27)$$

As the first step in order to fit the dimensions of (3.1) and of (3.25) the appropriate relation  $j = j(\Omega, L)$  has to be found and, as a consequence,  $m = m(m_\Omega, \Omega, L)$  [9]. Afterwards, we have to consider as the effective Hamiltonian the restriction of the matrix elements (3.26) and (3.27) to the invariant subspace spanned by  $|j, m\rangle$  with  $m = -j, -j + 2, \dots, j - 2, j$ . In this way we obtain the effective matrix elements

$$\langle j, m + 2 | H_q | j, m \rangle = \chi(q) h(L, \Omega, m_\Omega). \quad (3.28)$$

For values of  $L \geq \Omega$  ( $L$  integer), we find  $j = \Omega$ ,  $m = m_\Omega$  and the function  $h(L, \Omega, m_\Omega)$  is (the function  $\chi(q)$  is also explicitly defined in [9]):

$$h(L, \Omega, m_\Omega) = q^{2(m_\Omega+1)} \sqrt{[\Omega + m_\Omega + 2]_q [\Omega + m_\Omega + 1]_q [\Omega - m_\Omega]_q [\Omega - m_\Omega - 1]_q}. \quad (3.29)$$

Afterwards, we have followed the same procedure as in the  $q$ -DPS model, and we have looked for values of the  $q$  parameter (and, consequently, of  $\chi(q)$ ) which may absorb bosonic degrees of freedom of (3.1) and give rise to a similar spectrum for the purely fermionic  $q$ -deformed Hamiltonian (3.25). The second type of LE model (3.8) can also be approximated by the same type of  $q$ -Hamiltonian (3.25) and, in both cases, numerical computations lead to an excellent fitting between the LE models and the effective  $q$ -Hamiltonians [9]. Therefore, we conclude that certain interactions between fermions and bosons can accurately be described by using  $q$ -fermions as quasiparticles (i.e; effective fermionic degrees of freedom) under the exactly solvable interaction given by the Hamiltonian  $H_q^{\text{int}}$  (1.3).

From the mathematical point of view, a very interesting question to be solved is to find a suitable analytical expression of the deformation parameter  $q$  in terms of the representation space labels  $\Omega$  and  $L$ . In this respect, the quartic LE algebras previously introduced should be relevant, since the  $su_q(2)$  model (3.25) leads to the following natural definition of the  $K$  operators

$$K_0 = \tilde{T}_0 \quad K_+ = q^{\tilde{T}_0} \tilde{T}_+^2 q^{\tilde{T}_0} \quad K_- = q^{\tilde{T}_0} \tilde{T}_-^2 q^{\tilde{T}_0} \quad (3.30)$$

and their commutation rule  $[K_+, K_-]$  (which is a generalization of (2.28)) has to carry essentially the same algebraic information as (3.19) and (3.23). Another interesting feature appears in the analysis of the effective  $q$ -Hamiltonian for  $H_{++}$  (3.8), since a  $su(2)$  symmetry of the model can dynamically be restored for certain negative values of  $L$  [9]. A complete study of the LE algebras together with the abovementioned algebraic properties of these fermion-boson interactions will be addressed in a forthcoming paper [22].

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